

# BOHR'S PHENOMENON ON A REGULAR CONDENSATOR IN THE COMPLEX PLANE

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RÉSUMÉ. We prove the following generalisation of Bohr's theorem : let  $K \subset \mathbb{C}$  a continuum,  $(F_{K,n})_{n \geq 0}$  its Faber polynomials,  $\Omega_R$  the level sets of the Green function of  $\mathbb{C} \setminus K$  with singularity at infinity, then there exists  $R_0$  such that for any  $f = \sum_n a_n F_{K,n} \in \mathcal{O}(\Omega_{R_0}) : f(\Omega_{R_0}) \subset D(0, 1)$  implies  $\sum_n |a_n| \cdot \|F_{K,n}\|_K < 1$ .

## 1. INTRODUCTION

The well-known Bohr's theorem [2] states that for any function  $f(z) = \sum_{n \geq 0} a_n z^n$  holomorphic on the unit disc  $\mathbb{D}$  :

$$\left( \left| \sum_{n \geq 0} a_n z^n \right| < 1, \forall z \in \mathbb{D} \right) \implies \left( \sum_{n \geq 0} |a_n z^n| < 1, \forall z \in D(0, 1/3) \right)$$

and the constant  $1/3$  is optimal.

Our goal in this work is to study Bohr's theorem in the following context. Let  $K \subset \mathbb{C}$  be a compact in the complex plane. What are the open sets  $\Omega$  containing  $K$  such that the space  $\mathcal{O}(\Omega)$  admits a topological basis<sup>1</sup>  $(\varphi_n)_n$  which verifies, for every holomorphic function  $f = \sum_{n \geq 0} a_n \varphi_n \in \mathcal{O}(\Omega)$  :

$$\left( \left| \sum_{n \geq 0} a_n \varphi_n(z) \right| < 1, \forall z \in \Omega \right) \implies \left( \sum_{n \geq 0} |a_n| \cdot \|\varphi_n\|_K < 1 \right) ?$$

In this case we say that the family  $(K, \Omega, (\varphi_n)_{n \geq 0})$  satisfies **Bohr's property** or that **Bohr's phenomenon** is observed.

**Some examples :** • The family  $(\overline{D(0, 1/3)}, D(0, 1), (z^n)_{n \geq 0})$  satisfies Bohr's phenomenon (this is Bohr's classic theorem).

• Note that the family  $(\overline{D(0, 1/3)}, D(0, 1), ((3z)^n)_{n \geq 0})$  also satisfies Bohr's phenomenon. This example will play a special role in the following, since  $((3z)^n)_{n \geq 0}$  is the Faber polynomial basis associated with the compact  $\overline{D(0, 1/3)}$ .

• On the other hand, the family  $(\overline{D(0, 2/3)}, D(0, 1), (z^n)_{n \geq 0})$  does not satisfy Bohr's phenomenon (due to optimality of the constant  $1/3$  in Bohr's theorem).

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1. For all  $f \in \mathcal{O}(\Omega)$  there exists an unique sequence  $(a_n)_n$  of complex numbers such that  $f = \sum_{n \geq 0} a_n \varphi_n$  for the usual compact convergence topology of  $\mathcal{O}(\Omega)$ .

As a starting point, for a given compact  $K$  we must choose a “good” open neighborhood  $\Omega$ , that admits for  $\mathcal{O}(\Omega)$  a “nice” basis  $(\varphi_n)_n$ . “Nice” here means that there are good local estimates for  $\varphi_n$  on  $\Omega$  but not only, since, unlike for other well-known theorems for power series on the disc [7], Bohr’s theorem cannot be extended to all basis. For example, as pointed out by Aizenberg [1], it is necessary that one of the elements of the basis be a constant function.

We want to focus on the following situation :

**Définition 1.1.** *Let  $K$  be a compact in  $\mathbb{C}$  including at least two points,  $K$  is a continuum if  $\overline{\mathbb{C}} \setminus K$  is simply connected.*

When  $K$  is a continuum it can be associated with the sequence  $(F_{K,n})_n$  of its Faber polynomials. In more detail, let  $\Phi : \overline{\mathbb{C}} \setminus K \rightarrow \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  be the unique conformal mapping that verifies

$$\Phi(\infty) = \infty, \quad \Phi'(\infty) = \gamma > 0.$$

Therefore  $\Phi$  admits a Laurent development close to the infinity point under the form :

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \dots + \frac{\gamma_k}{z^k} + \dots$$

and then for  $n \in \mathbb{N}$  :

$$\begin{aligned} \Phi^n(z) &= \left( \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \dots + \frac{\gamma_k}{z^k} + \dots \right)^n \\ &= \underbrace{\gamma^n z^n + a_{n-1}^{(n)} z^{n-1} + \dots + a_1^{(n)} z + a_0^{(n)}}_{F_{K,n}(z)} + \underbrace{\frac{b_1^{(n)}}{z} + \frac{b_2^{(n)}}{z^2} + \dots + \frac{b_k^{(n)}}{z^k} + \dots}_{E_{K,n}(z)} \end{aligned}$$

$F_{K,n}$  is the polynomial part of the Laurent expansion at infinity of  $\Phi^n$ . It is a common basis for the spaces  $\mathcal{O}(K)$ ,  $\mathcal{O}(\Omega_R)$ , ( $R > 1$ ) where  $\Omega_R := \{z \in \mathbb{C} : |\Phi(z)| < R\} \cup K$ . This polynomial basis exhibits remarkable properties (the relevant reference is the work by P.K.Suetin [10]) similar to the Taylor basis  $(z^n)_n$  on discs  $D(0, R)$ . In particular, the level sets  $\Omega_R$  are the convergence domains of the series  $\sum_{n \geq 0} a_n F_{K,n}$  and for any compact  $L \subset \overline{\mathbb{C}} \setminus K$  we have

$$\lim_{n \rightarrow \infty} \|F_{K,n}\|_L^{1/n} = \|\Phi\|_L.$$

This formula is the one variable version of a more general formula (see [8]).

In this work, we show (Theorem 3.1) that for every continuum  $K$  there exists an  $R_0 > 1$  such that for any  $R \geq R_0$  the family  $(K, \Omega_R, (F_{K,n})_{n \geq 0})$  verifies Bohr’s property. We start by studying the cases of an elliptic condensator (i.e.  $K = [-1, 1]$ ) which had been considered in a different form by Kaptanoglu and Sadik in an interesting study [5] which motivated this article (see remark 2.4).

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2.  $\Omega_R$  is also the level set of the Siciak-Zaharjuta extremal function  $\Phi_K(z) := \sup\{|p(z)|^{1/\deg(p)}\}$  where the supremum is taken over all complex polynomials  $p$  such that  $\|p\|_K \leq 1$ .  $\Phi_K$  is also related to the classical Green function for  $\overline{\mathbb{C}} \setminus K$  with pole at infinity  $g_K : \mathbb{C} \setminus K \rightarrow ]0, +\infty[$  by the equality  $\log \Phi_K = g_K$  on  $\mathbb{C} \setminus K$ . Recall that  $g_K$  is the unique harmonic positive function on  $\mathbb{C} \setminus K$  such that  $\lim_{z \rightarrow \infty} (g_K(z) - \log |z|)$  exists and is finite and  $\lim_{z \rightarrow w} g_K(z) = 0$ ,  $\forall w \in \partial(\mathbb{C} \setminus K)$ .

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## 2. AN EXAMPLE : THE “ELLIPTIC” CONDENSATOR $K = [-1, 1]$

Let us examine in this section the particular case where  $K := [-1, 1]$ . This is a “fundamental” example because this is one of the very few case (see [10], [4] for circular lunes) where the explicit form of the conformal map  $\Phi : \Omega := \mathbb{C} \setminus K \rightarrow \{|w| > 1\}$  allows us to obtain a more precise estimation of the Faber polynomials of  $K$  (see [10]).

Here,  $\Phi^{-1}(w) = \frac{1}{2}(w + w^{-1})$  is the Zhukovskii function, the Faber polynomials  $(F_{K,n})_n$  form a common basis for the spaces  $\mathcal{O}(\Omega_R)$ , ( $R > 1$ ) where the boundary  $\partial\Omega_R = \Phi^{-1}(\{|w| = R\})$  of the level set  $\Omega_R$  is given by the equation :

$$2z = Re^{i\theta} + R^{-1}e^{-i\theta}.$$

Theses are ellipses with foci 1 et  $-1$  and eccentricity  $\varepsilon = \frac{2R}{1+R^2}$ . We observe that the polynomials  $F_{K,n}$  enjoy in the target coordinates “ $w$ ” a much more convenient form for computation than in the source coordinates “ $z$ ”. Indeed  $\Phi$  presents a simple pole at infinity which implies that  $\Phi^n + 1/\Phi^n$  et  $\Phi^n$  have the same principal part. We observe also that  $\Phi(z) = z + \sqrt{z^2 - 1}$ , which implies  $1/\Phi(z) = z - \sqrt{z^2 - 1}$ . From these last identities we can deduce<sup>3</sup> that  $1/\Phi^n + \Phi^n$  extends as a polynomial on  $\mathbb{C}$ . This is  $F_{K,n}$  and if we write  $F_{K,n}$  in the target coordinates “ $w$ ”, we get :

$$F_{K,n}(w) = w^n + w^{-n}.$$

This important equality will allow us to write any function  $f(z) = \sum_n a_n F_{K,n}(z)$ ,  $z \in \Omega_R$ , holomorphic on  $\Omega_R$  under the form

$$f(z) = f(\Phi^{-1}(w)) = \sum_n a_n F_{K,n}(\Phi^{-1}(w)) = \sum_n a_n (w^n + w^{-n}), \quad 1 < |w| < R,$$

and we shall often use this device from now on.

Now let us look at Bohr’s phenomenon for the elliptic condensator ( $K := [-1, 1]$ ,  $\Omega_R$ ,  $(F_{K,n})_{n \geq 0}$ ) given that  $R > R_0$  is large enough. Then next proposition is, in our particular case, the equivalent version of Caratheodory’s inequality.

**Proposition 2.1.** *Let  $f(w) = a_0 + \sum_1^\infty a_n(w^n + w^{-n}) \in \mathcal{O}(\{1 < |w| < R\})$ . Suppose that  $\operatorname{re}(f) > 0$ , then :*

$$|a_n| \leq \frac{2\operatorname{re}(a_0)}{R^n - R^{-n}}, \quad \forall n > 0.$$

**Proof :** Let  $1 < r < R$ , then for all  $n > 0$  we have

$$\begin{aligned} a_n r^{-n} &= \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} f(re^{i\theta}) d\theta, \\ \bar{a}_n r^n &= \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \bar{f}(re^{i\theta}) d\theta. \end{aligned}$$

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3. We can also deduce (see [10], pp. 36-37) that if  $K = [-1, 1]$ , then the Faber polynomials are the Tchebyshev polynomials of the first kind (up to a constant 2 if  $n \geq 1$ ) :  $F_{K,0}(z) = T_0(z)$ ,  $F_{K,n}(z) = 2T_n(z)$ , ( $n \geq 1$ ) where  $T_n(x) = \cos(n \arccos x)$ .

which easily gives (remember that  $\operatorname{re}(f) > 0$ ) :

$$|a_n| \cdot (r^n - r^{-n}) \leq |a_n r^{-n} + \bar{a}_n r^n| \leq \frac{1}{\pi} \int_0^{2\pi} \operatorname{re}(f(re^{i\theta})) d\theta = 2\operatorname{re}(a_0),$$

to get the expected result, (just let  $r$  tend to  $R$ ).  $\blacksquare$

**Lemme 2.2.** *Let  $f = a_0 + \sum_{n=1}^{\infty} a_n(w^n + w^{-n}) \in \mathcal{O}(\{1 < |w| < R\})$ . Suppose that  $|f| < 1$  and  $a_0 > 0$ , then<sup>4</sup> we have :*

$$|a_n| \leq \frac{2(1 - a_0)}{R^n - R^{-n}}.$$

**Proof :** This is classical : let  $g = 1 - f$ , then  $\operatorname{re}(g) > 0$  on  $\{1 < |w| < R\}$  and by prop. 2.1 :

$$|a_n| \leq \frac{2(1 - a_0)}{R^n - R^{-n}}. \quad \blacksquare$$

**Proposition 2.3.** *For all  $R \geq R_0 = 5.1284\dots$  the family  $(K := [-1, 1], \Omega_R, (F_{K,n})_{n \geq 0})$  satisfies Bohr's phenomenon ( $\Omega_{R_0}$  is the ellipse with eccentricity  $\varepsilon_0 = 0.3757\dots$ ).*

**Proof :** Let  $f = a_0 + \sum_{n=1}^{\infty} a_n F_{K,n} \in \mathcal{O}(\Omega_R)$  and suppose that  $|f| < 1$  on  $\Omega_R$ . In the variables “ $w$ ” :  $f(w) = a_0 + \sum_{n=1}^{\infty} a_n(w^n + w^{-n})$  on  $\{1 < |w| < R\}$  and up to a rotation (changing nothing by symmetry), we can suppose that  $a_0 \geq 0$ . Then by lemma 2.2 :

$$\begin{aligned} a_0 + \sum |a_n| \cdot \|F_{K,n}\|_K &\leq a_0 + 2(1 - a_0) \sum_{n=1}^{\infty} \frac{r^n + r^{-n}}{R^n - R^{-n}}, \quad (1 < r < R) \\ &\leq a_0 + (1 - a_0) \sum_{n=1}^{\infty} \frac{4R^n}{R^{2n} - 1}. \end{aligned}$$

This gives

$$a_0 + \sum_{n=1}^{\infty} |a_n| \cdot \|F_{K,n}\|_K < 1$$

if

$$\varphi(R) := \sum_{n=1}^{\infty} \frac{4R^n}{R^{2n} - 1} < 1.$$

But  $\varphi$  strictly decreases on  $]1, \infty[$ ,  $\lim_{1+} \varphi(R) = +\infty$ ,  $\lim_{+\infty} \varphi(R) = 0$  therefore, there exists a unique  $R_0 > 1$  such that  $\varphi(R) - 1 = 0$  on  $]1, \infty[$ ; Mathematica gives  $R_0 = 5.1284\dots$  corresponding to an eccentricity of  $\varepsilon_0 = 0.3757\dots$ ;  $(K := [-1, 1], \Omega_R, (F_{K,n})_{n \geq 0})$  satisfies Bohr's phenomenon for all  $R \geq R_0$ .  $\blacksquare$

**Remarque 2.4.** *Using theorem 7 in [5], we can deduce a weaker version of proposition 2.3 with  $R_0 = 5.1573\dots$  and  $\varepsilon_0 = 0.3738\dots$ , so, proposition 2.3 is a slightly stronger version of theorem 7 in [5]. In another work [6] we calculate exactly the infimum of  $R_0$  satisfying proposition 2.3 i.e. what we call the Bohr's radius of  $K = [-1, 1]$  in Theorem 3.1.*

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4. Note that  $|f| < 1$  implies  $a_0 < 1$ .

## 3. BOHR'S PHENOMENON ON AN ARBITRARY GREEN CONDENSATOR

**3.1. Estimations of Faber polynomials on a Green condensator.** In this paragraph, we recall classical inequalities (see [10]) on Faber polynomials of  $K$  that we will use in paragraph 3.2.

Let  $K \subset \mathbb{C}$  be a continuum,  $(F_{K,n})_{n \geq 0}$  its Faber polynomials. Recall that  $\Phi^n(z) = F_{K,n}(z) + E_{K,n}(z)$  where  $E_{K,n}$  is the meromorphic part in the Laurent developement of  $\Phi^n$  in a neighborhood of infinity. If  $\Omega_r$ , ( $r > 1$ ) is the level set  $\{z \in \mathbb{C} : |\Phi(z)| < r\}$  then we have the following integral formulas for Faber polynomials (see Suetin, [10], pp 42) :

$$(1) \quad \forall z \in \Omega_r : \quad F_{K,n}(z) = \int_{\partial\Omega_r} \frac{\Phi^n(t)}{t-z} dt,$$

$$(2) \quad \forall z \in \mathbb{C} \setminus \overline{\Omega_r} : \quad E_{K,n}(z) = \int_{\partial\Omega_r} \frac{\Phi^n(t)}{t-z} dt,$$

Formula (2) leads to the following estimations for all  $1 < r < R$  :

$$(3) \quad \forall z \in \mathbb{C} \setminus \Omega_R : \quad |E_{K,n}(z)| \leq \int_{\partial\Omega_r} \left| \frac{\Phi^n(t)}{t-z} \right| \cdot |dt| \leq \frac{r^n \text{lg}(\partial\Omega_r)}{\text{dist}(z, \partial\Omega_r)},$$

( $\text{lg}(\partial\Omega_r)$  is the euclidian length  $\partial\Omega_r$ ,  $\text{dist}(z, \partial\Omega_r)$  is the euclidian distance from  $z$  to  $\partial\Omega_r$ ) and

$$(4) \quad \forall z \in \partial\Omega_R : \quad |F_{K,n}(z)| \leq R^n \left( 1 + \frac{r^n}{R^n} \cdot \frac{\text{lg}(\partial\Omega_r)}{\text{dist}(z, \partial\Omega_r)} \right),$$

for all  $1 < r < R$ . Then if  $R$  is large enough, precisely if

$$\frac{r^n}{R^n} \cdot \frac{\text{lg}(\partial\Omega_r)}{\text{dist}(z, \partial\Omega_r)} < 1$$

then for all  $n > 0$ , we have :

$$\forall z \in \partial\Omega_R : \quad |F_{K,n}(z)| \geq R^n \left( 1 - \frac{r^n}{R^n} \cdot \frac{\text{lg}(\partial\Omega_r)}{\text{dist}(z, \partial\Omega_r)} \right) > 0.$$

With formula (1), we deduce the estimation, for all  $r > 1$  and  $z \in K$  :

$$(5) \quad |F_{K,n}(z)| \leq \int_{\partial\Omega_r} \left| \frac{\Phi^n(t)}{t-z} \right| \cdot |dt| \leq r^n \frac{\text{lg}(\partial\Omega_r)}{\text{dist}(z, \partial\Omega_r)}.$$

If moreover the compact  $K$  is a domain defined by a real analytic Jordan curve, then Caratheodory's theorem ensures that  $\Phi$  extends as a biholomorphism on a neighborhood of  $\partial K$ , say up to  $\partial\Omega_{r_0}$ , where  $r_0 < 1$ . From this, we get for all  $r_0 < R$  :

$$\forall z \in \mathbb{C} \setminus \Omega_R : \quad |E_{K,n}(z)| \leq \int_{\partial\Omega_{r_0}} \left| \frac{\Phi^n(t)}{t-z} \right| \cdot |dt| \leq \frac{r_0^n \text{lg}(\partial\Omega_{r_0})}{\text{dist}(z, \partial\Omega_{r_0})},$$

and so the estimations

$$\begin{aligned} R^n \left( 1 - \frac{r_0^n}{R^n} \cdot \frac{\text{lg}(\partial\Omega_{r_0})}{\text{dist}(z, \partial\Omega_{r_0})} \right) &\leq |F_{K,n}(z)| \\ &\leq R^n \left( 1 + \frac{r_0^n}{R^n} \cdot \frac{\text{lg}(\partial\Omega_{r_0})}{\text{dist}(z, \partial\Omega_{r_0})} \right), \end{aligned}$$

for all  $z \in \mathbb{C} \setminus \Omega_R$ ,  $r_0 < R$ .

**3.2. Bohr's phenomenon on a Green condensor.** In this paragraph we extend proposition 2.3 for all continuum  $K$  in the complex plane, precisely :

**Théorème 3.1.** *For all continuum  $K \subset \mathbb{C}$ , there exists a constant  $R_K > 1$  such that for all  $R > R_K$  the family  $(K, \Omega_R, (F_{K,n})_{n \geq 0})$  satisfies Bohr's phenomenon and the infimum  $R_0$  of such  $R$  will be called the **Bohr's radius** of  $K$ .*

For example the Bohr radius for a disc  $K = D(a, r)$  is 3 due to Bohr's classic theorem, and in [6] we compute the exact value of  $R_0$  when  $K = [-1, 1]$ .

Before proving theorem 3.1, some intermediate results are necessary. Let  $K$  be a continuum,  $(F_{K,n})_{n \geq 0}$  its sequence of Faber polynomials and  $z_0 \in \partial K$ . Consider the family  $(\varphi_n)_{n \geq 0}$  where  $\varphi_0 \equiv 1$  and  $\varphi_n = F_{K,n} - F_{K,n}(z_0)$  ( $n \geq 1$ ). It is clear that  $(\varphi_n)_{n \geq 0}$  is again a basis of the spaces  $\mathcal{O}(\Omega_R)$  for all  $R > 1$  and we have

**Théorème 3.2.** *The family  $(K, \Omega_R, (\varphi_n)_{n \geq 0})$  enjoys Bohr's property for  $R$  large enough. That is to say, there exists  $R > 1$  such that all holomorphic function  $f = \sum_n a_n \varphi_n \in \mathcal{O}(\Omega_R)$  with values in  $\mathbb{D}$  satisfy*

$$\sum_{n \geq 0} |a_n| \cdot \|\varphi_n\|_K = |f(z_0)| + \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K < 1.$$

**Proof :** Let  $R_0 > 1$ . We can suppose without loss of generality that  $z_0 = 0$ . Because  $\varphi_n(0) = 0$  for all  $n > 0$  we can apply theorem 3.3 in [1] on the open set  $\Omega_{R_0}$ . This implies that there exists  $D(0, \rho_0)$  where  $\rho_0$  is small enough and a compact  $K_1 \subset \Omega_{R_0}$  such that :

$$|f(0)| + \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_{D(0, \rho_0)} \leq \|f\|_{K_1},$$

for any function  $f = \sum_n a_n \varphi_n \in \mathcal{O}(\Omega_{R_0})$ . Now choose  $\rho_1 > 0$  such that  $K_1 \subset D(0, \rho_1)$ . We have :

$$(6) \quad |f(0)| + \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_{D(0, \rho_0)} \leq \|f\|_{D(0, \rho_1)},$$

for all  $f = \sum_n a_n \varphi_n \in \mathcal{O}(\Omega_R)$  where  $R$  is choosen large enough so that  $D(0, \rho_1) \subset \Omega_R$ .

Let  $f \in \mathcal{O}(\Omega_R)$  such that  $\|f\|_{\Omega_R} \leq 1$ ; the invariant form of Schwarz's lemma ([3], chapter 8) gives the following estimation on any disc  $D(0, \rho) \subset \Omega_R$  ( $\rho \geq \rho_1$ ) :

$$(7) \quad \|f\|_{D(0, \rho_1)} \leq \frac{\rho_1 \rho^{-1} + |f(0)|}{1 + |f(0)| \rho_1 \rho^{-1}}.$$

We want for  $f = f(0) + \sum_{n \geq 1} a_n \varphi_n \in \mathcal{O}(\Omega_R)$  to dominate the quantity :  $|f(0)| + \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K$ ; write

$$(8) \quad \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K = \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_{D(0, \rho_0)} \times \frac{\|\varphi_n\|_K}{\|\varphi_n\|_{D(0, \rho_0)}}.$$

Let  $L$  be a disc contained in  $D(0, \rho_0) \setminus K$  then

$$\lim_{n \rightarrow \infty} \|\varphi_n\|_L^{1/n} = R^{\alpha_L}$$

where<sup>5</sup>  $\alpha_L := \max_{z \in L} \omega(z, K, \Omega_R)$ , this is in fact true for all compact  $L \subset \Omega_R \setminus K$  and this is an immediate corollary of a Nguyen Thanh Van's result ([8], page 228, see also [9], [11] for “pluricomplex versions”). At this point, it's not difficult to deduce

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0 : \|\varphi_n\|_K \leq C_\varepsilon R^{n\varepsilon}, \forall n \in \mathbb{N},$$

and

$$\exists C > 0 : \|\varphi_n\|_L \geq C \cdot R^{n\frac{\alpha_L}{2}}, \forall n \in \mathbb{N}.$$

It remains to choose  $\varepsilon > 0$  small enough so that  $R^\varepsilon < R^{\frac{\alpha_L}{2}}$ . Such a choice assures

$$0 \leq \lim_{n \rightarrow +\infty} \frac{\|\varphi_n\|_K}{\|\varphi_n\|_{D(0, \rho_0)}} \leq \lim_{n \rightarrow +\infty} \left(R^{\varepsilon - \frac{\alpha_L}{2}}\right)^n = 0.$$

So the sequence  $\left(\frac{\|\varphi_n\|_K}{\|\varphi_n\|_{D(0, \rho_0)}}\right)_n$  is bounded : by (8) there exists  $C > 0$  such that

$$\sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K \leq C \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_{D(0, \rho_0)},$$

which give us with (6) the estimation :

$$\sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K \leq C(\|f\|_{D(0, \rho_1)} - |f(0)|).$$

Finally, with the invariant Schwarz's lemma (7)

$$\sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K \leq C \left( \frac{\rho_1 \rho^{-1} + |f(0)|}{1 + |f(0)| \rho_1 \rho^{-1}} - |f(0)| \right) = C \rho_1 \rho^{-1} \left( \frac{1 - |f(0)|^2}{1 + |f(0)| \rho_1 \rho^{-1}} \right)$$

which lead us to the main estimation

$$\sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K \leq 2C \rho_1 \rho^{-1} (1 - |f(0)|).$$

To conclude, let us choose  $\rho$  large enough so that  $2C \rho_1 \rho^{-1} \leq 1$ , therefore, for any  $R > 1$  such that  $D(0, \rho) \subset \Omega_R$  and  $f = f(0) + \sum_{n \geq 1} a_n \varphi_n \in \mathcal{O}(\Omega_R)$ ,  $f(\Omega_R) \subset \mathbb{D}$ , we have :

$$|f(0)| + \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K \leq 1.$$

Q.E.D. ■

Of course we must now come back to the basis  $(F_{K,n})_n$  :

**Lemme 3.3.** *Let  $\tilde{K} \subset K$  be another compact,  $(\varepsilon_n)_{n \geq 1}$  a complex sequence and suppose that there exists a constant  $0 < C < 1$  such that*

$$\left\{ \sup_{z \in \tilde{K}} |\varphi_n(z) - \varepsilon_n| \leq C \cdot \|\varphi_n\|_K, \quad \forall n \in \mathbb{N}, \right. \quad (9)$$

$$\left. |\varepsilon_n| \leq (1 - C) \cdot \|\varphi_n\|_K, \quad \forall n \in \mathbb{N}. \right. \quad (10)$$

*Then the family  $(\tilde{K}, \Omega, (\tilde{\varphi}_n)_{n \geq 0})$  satisfies Bohr's property with  $\tilde{\varphi}_0 \equiv 1$ ,  $\tilde{\varphi}_n := \varphi_n - \varepsilon_n$ .*

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5.  $\omega$  is the extremal function associated for the pair  $(K, \Omega_R)$ .

**Proof :** Let  $f = a_0 + \sum_{n \geq 1} a_n \varphi_n = a_0 + \sum_{n \geq 1} a_n \epsilon_n + \sum_{n \geq 1} a_n (\varphi_n - \epsilon_n) \in \mathcal{O}(\Omega)$  and suppose that  $|f| \leq 1$  on  $\Omega$ . We have to prove that

$$\left| a_0 + \sum_{n \geq 1} a_n \epsilon_n \right| + \sum_{n \geq 1} |a_n| \cdot \|\varphi_n - \epsilon_n\|_{\tilde{K}} \leq 1.$$

But :

$$\begin{aligned} \left| a_0 + \sum_{n \geq 1} a_n \epsilon_n \right| + \sum_{n \geq 1} |a_n| \cdot \|\varphi_n - \epsilon_n\|_{\tilde{K}} &\leq \\ &\leq \left| a_0 + \sum_{n \geq 1} a_n \epsilon_n \right| + C \cdot \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K \\ &\leq |a_0| + \sum_{n \geq 1} |a_n| \cdot |\epsilon_n| + C \cdot \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K \\ &\leq |a_0| + (1 - C) \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K + C \cdot \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K \\ &\leq |a_0| + \sum_{n \geq 1} |a_n| \cdot \|\varphi_n\|_K \leq 1 \end{aligned}$$

Q.E.D. ■

**Proof of Theorem 3.1 :** Now let  $K$  be a continuum,  $\Omega_R$ , ( $R > 1$ ), a level set of the Green function of  $K$  and fix  $\tilde{K} = \overline{\Omega_R}$ . If  $a \in \partial\Omega_R$  there exists (this is theorem 3.2)  $R' > R$  such that the family  $(\overline{\Omega_R}, \Omega_{R'}, (1, F_{\tilde{K},n} - F_{\tilde{K},n}(a))_{n \geq 0})$  satisfies Bohr's property. Then for any function

$$f = a_0 + \sum_{n \geq 1} a_n (F_{\tilde{K},n} - F_{\tilde{K},n}(a)) \in \mathcal{O}(\Omega_{R'}),$$

such that  $|f| \leq 1$  on  $\Omega_{R'}$ , we have

$$|a_0| + \sum_{n \geq 1} |a_n| \cdot \|F_{\tilde{K},n} - F_{\tilde{K},n}(a)\|_{\overline{\Omega_R}} \leq 1.$$

But ([10], page 35) :  $F_{\tilde{K},n}(z) = R^{-n} F_{K,n}(z)$  so

$$\begin{aligned} f(z) &= a_0 + \sum_{n \geq 1} a_n (F_{\tilde{K},n}(z) - F_{\tilde{K},n}(a)) \\ &= f(z) = a_0 + \sum_{n \geq 1} a_n R^{-n} (F_{K,n}(z) - F_{K,n}(a)). \end{aligned}$$

Because  $R > 1$ , this immediately implies that the basis  $(1, F_{K,n} - F_{K,n}(a))_{n \geq 0}$  satisfies Bohr's property on  $(\overline{\Omega_R}, \Omega_{R'})$ . If we apply lemma 3.3 with  $\varphi_n = F_{K,n} - F_{K,n}(a)$ ,  $a \in \partial\Omega_R$  and  $-\epsilon_n = F_{K,n}(a)$ , the inequalities (9) and (10) are :

$$(9') \quad \sup_{z \in K} |F_{K,n}(z)| \leq C \cdot \sup_{z \in \overline{\Omega_R}} |F_{K,n}(z) - F_{K,n}(a)|,$$

$$(10') \quad |F_{K,n}(a)| \leq (1 - C) \cdot \sup_{z \in \overline{\Omega_R}} |F_{K,n}(z) - F_{K,n}(a)|.$$



(where  $C \in ]0, 1[$  is a constant). For all  $n \in \mathbb{N}$  choose  $a_n \in \partial\Omega_R$  such that  $\Phi(a_n) = \theta_n \Phi(a)$  where  $\theta_n$  is an  $n$ -root of  $-1$  (remember that  $\Phi(\partial\Omega_R) = C(0, R)$ ). So

$$F_{K,n}^n(a_n) = \Phi^n(a_n) - E_{K,n}(a_n) = -\Phi(a)^n - E_{K,n}(a_n)$$

and

$$F_{K,n}(a_n) - F_{K,n}(a) = -2\Phi(a)^n - [E_{K,n}(a) + E_{K,n}(a_n)].$$

But because of inequality (3) in paragraph 3.1, noting  $r = 1 + \varepsilon_0$  :

$$|E_{K,n}(a) + E_{K,n}(a_n)| \leq 2(1 + \varepsilon_0)^n \frac{\lg(\partial\Omega_{1+\varepsilon_0})}{\text{dist}(\partial\Omega_{1+\varepsilon_0}, \partial\Omega_R)},$$

for all  $n \in \mathbb{N}$  and  $R > 1 + \varepsilon_0$ . Consequently :

$$\begin{aligned} \sup_{z \in \overline{\Omega_R}} |F_{K,n}(z) - F_{K,n}(a)| &\geq |F_{K,n}(a_n) - F_{K,n}(a)| \\ &\geq 2R^n \left[ 1 - \left( \frac{1 + \varepsilon_0}{R} \right)^n \cdot \frac{\lg(\partial\Omega_{1+\varepsilon_0})}{\text{dist}(\partial\Omega_{1+\varepsilon_0}, \partial\Omega_R)} \right] \end{aligned}$$

for all  $n \in \mathbb{N}$  and  $R > 1 + \varepsilon_0$ . So, as long as we choose  $R$  large enough, say  $R > R_0$ , we can suppose that

$$(11) \quad \sup_{z \in \overline{\Omega_R}} |F_{K,n}(z) - F_{K,n}(a)| \geq \frac{3}{2} R^n, \quad \forall n \in \mathbb{N}, R > R_0.$$

Because of (4) :

$$|F_{K,n}(a)| \leq R^n \left[ 1 + \left( \frac{1 + \varepsilon_0}{R} \right)^n \cdot \frac{\lg(\partial\Omega_{1+\varepsilon_0})}{\text{dist}(\partial\Omega_{1+\varepsilon_0}, \partial\Omega_R)} \right]$$

for all  $n \in \mathbb{N}$ ,  $R > 1 + \varepsilon_0$ . Because the term in between the brackets satisfies :

$$\begin{aligned} 1 &\leq 1 + \left( \frac{1 + \varepsilon_0}{R} \right)^n \cdot \frac{\lg(\partial\Omega_{1+\varepsilon_0})}{\text{dist}(\partial\Omega_{1+\varepsilon_0}, \partial\Omega_R)} \\ &\leq 1 + \left( \frac{1 + \varepsilon_0}{R_1} \right)^n \cdot \frac{\lg(\partial\Omega_{1+\varepsilon_0})}{\text{dist}(\partial\Omega_{1+\varepsilon_0}, \partial\Omega_{R_1})} \xrightarrow{R_1 \rightarrow \infty} 1 \end{aligned}$$

for all  $R > R_1 > 1 + \varepsilon_0$ ; it is less than  $5/4$  for all  $n \in \mathbb{N}$  and  $R > R_1$  where  $R_1$  is choosen large enough; i.e.

$$|F_{K,n}(a)| \leq \frac{5}{4} \cdot R^n, \quad \forall n \in \mathbb{N}, R > R_1.$$

It follows from (11) that

$$|F_{K,n}(a)| \leq \frac{5}{6} \cdot \frac{3}{2} \cdot R^n \leq \frac{5}{6} \sup_{z \in \overline{\Omega_R}} |F_{K,n}(z) - F_{K,n}(a)|, \quad \forall n \in \mathbb{N}, R > R_2 := \max\{R_0, R_1\}.$$

So we have proved inequality (10') with  $C = 1/6$ . Finally, still because of (4) :

$$\begin{aligned} \sup_{z \in K} |F_{K,n}(z)| &\leq \sup_{z \in \overline{\Omega_{1+2\varepsilon_0}}} |F_{K,n}(z)| \\ &\leq (1 + 2\varepsilon_0)^n \cdot \left[ 1 + \left( \frac{1 + \varepsilon_0}{1 + \varepsilon_0} \right)^n \cdot \frac{\lg(\partial\Omega_{1+\varepsilon_0})}{\text{dist}(\partial\Omega_{1+2\varepsilon_0}, \partial\Omega_{1+\varepsilon_0})} \right] \\ &\leq A(1 + 2\varepsilon_0)^n, \quad \forall n \in \mathbb{N} \end{aligned}$$

where  $A$  is a constant strictly larger than 1. Given  $A > 1$  being fixed it is easy to deduce that for any  $R > R_3$  :

$$\sup_{z \in K} |F_{K,n}(z)| \leq A(1 + 2\varepsilon_0)^n \leq \frac{R^n}{4}, \quad \forall n \in \mathbb{N}.$$

So because of (11)

$$\sup_{z \in K} |F_{K,n}(z)| \leq \frac{1}{6} \cdot \frac{3}{2} R^n \leq \frac{1}{6} \sup_{z \in \overline{\Omega}_R} |F_{K,n}(z) - F_{K,n}(a)|$$

for all  $n \in \mathbb{N}$  et  $R > \max\{R_3, R_2\}$ . This is formula (9') with  $C = 1/6$ , so we can apply lemma 3.3 and deduce that the family  $(K, \Omega_R, (F_{K,n})_{n \geq 0})$  satisfies Bohr's phenomenon for all  $R$  large enough : theorem 3.1. is proved. ■

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